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ON THE ADAPTIVE CONTROL OF JUMP PARAMETER SYSTEMS VIA NONLINEAR FILTERING*

PETER E. CAINES† AND JI-FENG ZHANG‡

Abstract. In this paper we first present an error analysis for the process of estimates generated by the Wonham filter when it is used for the estimation of the (finite set-valued) jump-Markov parameters of a random parameter linear stochastic system and further give bounds on certain functions of these estimates. We then consider a certainty equivalence adaptive linear-quadratic Gaussian feedback control law using the estimates generated by the nonlinear filter and demonstrate the global existence of solutions to the resulting closed-loop system. A stochastic Lyapunov analysis establishes that the certainty equivalence law stabilizes the Markov jump parameter linear system in the mean square average sense. The conditions for this result are that certain products of (i) the parameter process jump rate and (ii) the solution of the control Riccati equation and its second derivatives should be less than certain given bounds. An example is given where the controlled linear system has state dimension 2. Finally, the stabilizing properties of certainty equivalence laws which depend on (i) the maximum likelihood estimate of the parameter value and (ii) a modified version of this estimate are established under certain conditions.

Key words. jump parameter, nonlinear filter, adaptive control, stochastic systems, maximum likelihood

AMS subject classifications. 93E11, 93E15, 93E35

1. Introduction.

The hybrid system considered in this work is taken to have the following form:

$$dx_t = [A(\theta_t)x_t + B(\theta_t)u_t]dt + dw_t, \tag{1.1}$$

where $x_t \in \mathbb{R}^n$ and $u_t \in \mathbb{R}^m$ are the state and input of the system, $\{w_t, \mathcal{F}_t\}$ is a standard Wiener process in $\mathbb{R}^n$ with respect to a probability space $(\Omega, P, \mathcal{F})$, and $\theta_t \in \{1, 2, \ldots, N\}$ is the N-state jump-Markov parameter process subject to

$$\Phi_t = \Phi_0 + \Pi \int_0^t \Phi_s ds + m_t. \tag{1.2}$$

Here, $\Phi_t = [\mathbb{1}_{\{\theta_t=1\}}, \mathbb{1}_{\{\theta_t=2\}}, \ldots, \mathbb{1}_{\{\theta_t=N\}}]^\tau$ is the indicator process for θ_t, Π is the transition probability rate matrix, m_t is a zero-mean L^2 martingale, measurable with respect to an increasing σ-field $\mathcal{F}_t$. Φ_0 is $\mathcal{F}_0$-measurable and $E\Phi_0 = p_0$.

For $\theta = i$, $A(\theta) = A_i$, and $B(\theta) = B_i$, where the A_i's and B_i's are, respectively, $\mathbb{R}^{n\times n}$ and $\mathbb{R}^{n\times m}$ matrices such that $\|A_i - A_j\| + \|B_i - B_j\| \neq 0$ for $i \neq j$. Here and hereafter, $\|X\| \triangleq [\lambda_{\max}(X^\tau X)]^{1/2}$, where $\lambda_{\max}(A)$ denotes the largest eigenvalue of a matrix A.

The model (1.1), (1.2) is particularly appropriate for the analysis of the control of time varying systems, since (1.1) has a variable structure. As indicated by the dependence of all matrix parameters on the indicator process Φ_t, it can be used as

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† Department of Electrical Engineering, McGill University, 3480 University Street, Montreal H3A 2A7, Canada, and Canadian Institute for Advanced Research.

‡ Department of Electrical Engineering, McGill University, 3480 University Street, Montreal H3A 2A7, Canada, and Institute of Systems Science, Academia Sinica, Beijing 100080, People's Republic of China.

a model for systems subject to random failures and structural changes. Moreover, (1.2) is a general model for jump-Markov parameter processes (see, e.g., Liptser and Shiryaev (1977)).

Control problems for such systems in a nonadaptive setting have been the subject of considerable theoretical research for the past two decades and Sworder and Chou (1985) and Ezzine and Haddad (1989) have given surveys of previous work on this topic.

Generally speaking, the previous works can be classified into three groups: one group (see, e.g., Sworder and Chou (1985); Ezzine and Haddad (1989); Mariton and Bertrand (1985); Mariton (1986); Ji and Chizeck (1990); Feng, Loparo, Ji, and Chizeck (1992)) deals with the case where the system state process x and the jump parameter process Φ can be observed completely at any time instant. The second group (see, e.g., Wonham (1965), Rishel (1981), Caines and Chen (1985), Chen and Caines (1989), Helmes and Rishel (1990), Caines and Nassiri-Toussi (1991)) is concerned with the adaptive case where the system state process x can be observed, but the jump parameter process Φ cannot be directly observed and is consequently estimated. This may, for instance, be carried out by an application of the Wonham filter (see, e.g., Caines and Chen (1985), Chen and Caines (1989), Caines and Nassiri-Toussi (1991)). The third group (see, e.g., Sworder (1991)) discusses the adaptive case where neither the system state process x nor the jump parameter process Φ can be observed.

Among the first group, it is worth mentioning that Ji and Chizeck (1990) and Feng, Loparo, Ji, and Chizeck (1992) examine the relationship between appropriately defined controllability and stabilizability properties, and establish necessary and sufficient conditions for (i) system stabilization and (ii) infinite time jump linear quadratic (JLQ) optimal controls to exist. However, in most situations, direct observation of system parameters is impossible and this leads to the use of adaptive control. Caines and Chen (1985) used the Wonham filter and a dynamic programming approach to obtain a finite-horizon adaptive optimal control law for a general jump-Markov system. In a continuation of this work, Caines and Nassiri-Toussi (1991) and Nassiri-Toussi and Caines (1991) carried out a stochastic Lyapunov analysis of a certainty equivalence stabilizing control law and gave an analysis of the resulting ergodic behavior of the system. It is shown that, under rather strong conditions on the magnitude of the jumps of the parameters and the rate of the jump parameter process, a certainty equivalence linear feedback regulator (using the parameter estimates generated by the Wonham filter) gives rise to stable ergodic behavior of the system (1.1), (1.2). In some special cases, where the system is deterministic or where indirect observations of the parameter are available, special solutions to this problem have also been given in Sworder and Chou (1985), while the general adaptive control problem for stochastic jump-Markov parameter systems is addressed in Rishel (1981), Caines and Chen (1985), Chen and Caines (1989), Helmes and Rishel (1990), Sworder (1991), Caines and Nassiri-Toussi (1991), Nassiri-Toussi and Caines (1991), and Dufour and Bertrand (1993). It should be remarked that Rishel (1981) was the first to use the Wonham filter to find the equations of the optimal linear quadratic Gaussian (LQG) controller for a system depending upon a (constant in time) unobserved finite set-valued random variable. More recently, Helmes and Rishel (1990) have given an explicit solution to this problem for the case of minimizing the expectation of the quadratic state deviation at a final time plus the integrated square of the control action. Sworder (1991) presents an approximation to the quadratic-optimal regulator problem for a situation in which there is an unconventional measurement architecture; the solution is in a form quite similar to that obtained in the complete observation case, but the gain equation is made more complicated by the presence of noise. Finally, in a recent paper, Dufour and Bertrand (1993) responded to an announcement (Caines and Zhang (1992)) of the results of the present paper by giving a form of averaged control law (with respect to the conditional densities) that adaptively stabilizes the jump parameter system in question whenever it satisfies a simple set of algebraic sufficient conditions.

The object of this paper is to establish the existence of stabilizing adaptive feedback controllers for jump parameter systems under relatively weak conditions.

In §2 of this paper, the Wonham filter for estimating the indicator process Φ from observations on x and u is presented, and the error behavior of the filter is analyzed. Theorem 2.1 gives a formula for the mean square estimation error of Φ and Corollaries 2.1 and 2.2 give bounds for the expectation of certain weighted integrals of the estimates; these are required in the subsequent stability analysis. Section 3 contains the principal adaptive control result of the paper. By use of a stochastic Lyapunov technique it is shown that an adaptive LQG certainty equivalence feedback control law, which employs parameter estimates generated by the nonlinear filter, stabilizes the system in an average mean square sense. This result is subject to the condition that (i) the rate of the jump process of the system and (ii) the magnitude of the solution to the control Riccati equation and its second derivative are such that two products of these quantities fall below specified bounds (see (3.8)). It is to be noted that there is no condition on the size of the jumps of the parameters. In §4, a nontrivial example of this theory is given concerning the adaptive control of a two-dimensional linear system with jump-Markov system matrices $\{A_i,\ 1 \le i \le N\}$. Finally, in §5, the stabilizing properties of certainty equivalence laws which depend on (i) the maximum likelihood estimate of the parameter value and (ii) a modified version of this estimate are established under certain conditions.

2. The nonlinear filter and preliminary results. Suppose that (i) A_i and B_i are known for $i = 1, \ldots, N$, (ii) $E\|x_0\|^2 < \infty$, (iii) the cross quadratic variation of m and w, i.e., $d\langle m, w\rangle_t/dt \equiv 0$, and (iv) u_t is an m-dimensional $\mathcal{F}_t^x \triangleq \sigma\{x_s,\ s \le t\}$-measurable control process. Set

$$\widehat{\Phi}_t = [\widehat{\Phi}_t(1), \ldots, \widehat{\Phi}_t(N)]^\tau \triangleq E(\Phi_t|\mathcal{F}_t^x), \quad \forall t \ge 0, \tag{2.1}$$

$$H_t = [A_1 x_t + B_1 u_t, \ldots, A_N x_t + B_N u_t], \tag{2.2}$$

and

$$\mathrm{Diag}\widehat{\Phi}_t = \begin{bmatrix} \widehat{\Phi}_t(1) & & 0 \\ & \ddots & \\ 0 & & \widehat{\Phi}_t(N) \end{bmatrix}. \tag{2.3}$$

Then the nonlinear Wonham filter for the values of the parameter indicator process Φ_t is given by (see, e.g., Chen and Caines (1989))

$$d\widehat{\Phi}_t = \Pi \widehat{\Phi}_t dt + (\mathrm{Diag}\widehat{\Phi}_t - \widehat{\Phi}_t \widehat{\Phi}_t^\tau) H_t^\tau d\overline{w}_t, \tag{2.4}$$

where $\{\overline{w}_t,\ \mathcal{F}_t^x\}$ is the Wiener process of innovations defined by the innovation representation of x_t:

$$d\overline{w}_t = dx_t - H_t \widehat{\Phi}_t dt. \tag{2.5}$$

THEOREM 2.1. *The conditional mean square estimation error of the filter* (2.4) *for the system* (1.1) *satisfies*

$$E\|\widetilde{\Phi}_t\|^2 = E\|\widetilde{\Phi}_0\|^2 + 2E\int_0^t \widetilde{\Phi}_s^\tau \Pi \widetilde{\Phi}_s ds - 2E\int_0^t \Phi_s^\tau \Pi \Phi_s ds$$

$$-E\int_0^t \mathrm{Tr}\left([\widehat{\Phi}_s\widehat{\Phi}_s^\tau - \mathrm{Diag}\widehat{\Phi}_s](H_s^\tau H_s)[\widehat{\Phi}_s\widehat{\Phi}_s^\tau - \mathrm{Diag}\widehat{\Phi}_s]\right) ds, \tag{2.6}$$

where $\widetilde{\Phi}_t \triangleq \Phi_t - \widehat{\Phi}_t$, *and* $\mathrm{Tr}(X)$ *denotes the trace of matrix* X.

Proof. By (2.2), (1.1) can be rewritten as

$$dx_t = H_t \Phi_t dt + dw_t,$$

which together with (2.5) results in

$$d\overline{w}_t = H_t \widetilde{\Phi}_t dt + dw_t.$$

Therefore, by (1.2) and (2.4), we have

$$\begin{aligned} d\widetilde{\Phi}_t &= \Pi \widetilde{\Phi}_t dt + [\widehat{\Phi}_t\widehat{\Phi}_t^\tau - \mathrm{Diag}\widehat{\Phi}_t]H_t^\tau d\overline{w}_t + dm_t \\ &= \Pi \widetilde{\Phi}_t dt + [\widehat{\Phi}_t\widehat{\Phi}_t^\tau - \mathrm{Diag}\widehat{\Phi}_t]H_t^\tau H_t \widetilde{\Phi}_t dt \\ &\quad + [\widehat{\Phi}_t\widehat{\Phi}_t^\tau - \mathrm{Diag}\widehat{\Phi}_t]H_t^\tau dw_t + dm_t, \end{aligned}$$

which combined with Ito's formula (see, e.g., Schwartz (1984)) leads to

$$\begin{aligned} \widetilde{\Phi}_t^\tau \widetilde{\Phi}_t &= \widetilde{\Phi}_0^\tau \widetilde{\Phi}_0 + 2\int_0^t \widetilde{\Phi}_s^\tau \Pi \widetilde{\Phi}_s ds \\ &\quad + 2\int_0^t \widetilde{\Phi}_s^\tau (\widehat{\Phi}_s\widehat{\Phi}_s^\tau - \mathrm{Diag}\widehat{\Phi}_s) H_s^\tau H_s \widetilde{\Phi}_s ds \\ &\quad + 2\int_0^t \widetilde{\Phi}_s^\tau [\widehat{\Phi}_s\widehat{\Phi}_s^\tau - \mathrm{Diag}\widehat{\Phi}_s] H_s^\tau dw_s + 2\int_0^t \widetilde{\Phi}_s^\tau dm_s \\ &\quad + \int_0^t \mathrm{Tr}\left([\widehat{\Phi}_s\widehat{\Phi}_s^\tau - \mathrm{Diag}\widehat{\Phi}_s](H_s^\tau H_s)[\widehat{\Phi}_s\widehat{\Phi}_s^\tau - \mathrm{Diag}\widehat{\Phi}_s]\right) ds \\ &\quad + \sum_{0<s\le t} (\widetilde{\Phi}_s - \widetilde{\Phi}_{s-})^\tau (\widetilde{\Phi}_s - \widetilde{\Phi}_{s-}). \end{aligned} \tag{2.7}$$

Since $\widehat{\Phi}_t$, as a solution of (2.4), is continuous, $(\widetilde{\Phi}_s - \widetilde{\Phi}_{s-}) = \Phi_s - \Phi_{s-}$. From this we see that

$$\sum_{0<s\le t} (\widetilde{\Phi}_s - \widetilde{\Phi}_{s-})^\tau (\widetilde{\Phi}_s - \widetilde{\Phi}_{s-}) = \sum_{0<s\le t} (\Phi_s - \Phi_{s-})^\tau (\Phi_s - \Phi_{s-}) = 2J_t, \tag{2.8}$$

where J_t is the number of the jump points of Φ_s in $[0,t]$.

Substituting (2.8) into (2.7) and taking expectations on both sides, we see that

$$\begin{aligned} E\widetilde{\Phi}_t^\tau \widetilde{\Phi}_t &= E\widetilde{\Phi}_0^\tau \widetilde{\Phi}_0 + 2E\int_0^t \widetilde{\Phi}_s^\tau \Pi \widetilde{\Phi}_s ds + 2EJ_t \\ &\quad + 2E\int_0^t \widetilde{\Phi}_s^\tau (\widehat{\Phi}_s\widehat{\Phi}_s^\tau - \mathrm{Diag}\widehat{\Phi}_s) H_s^\tau H_s \widetilde{\Phi}_s ds \\ &\quad + E\int_0^t \mathrm{Tr}\left([\widehat{\Phi}_s\widehat{\Phi}_s^\tau - \mathrm{Diag}\widehat{\Phi}_s](H_s^\tau H_s)[\widehat{\Phi}_s\widehat{\Phi}_s^\tau - \mathrm{Diag}\widehat{\Phi}_s]\right) ds. \end{aligned} \tag{2.9}$$

From (1.2) and Ito's formula it follows that

$$\Phi_t^\tau \Phi_t = \Phi_0^\tau \Phi_0 + 2\int_0^t \Phi_s^\tau \Pi \Phi_s ds + 2\int_0^t \Phi_s^\tau dm_s + 2J_t,$$

which, together with $\Phi_t^\tau \Phi_t = \Phi_0^\tau \Phi_0 = 1$, implies

$$EJ_t = -E\int_0^t \Phi_s^\tau \Pi \Phi_s ds. \tag{2.10}$$

Notice that

$$E(\widetilde{\Phi}_t \widetilde{\Phi}_t^\tau | \mathcal{F}_t^x) = E(\Phi_t \Phi_t^\tau | \mathcal{F}_t^x) - \widehat{\Phi}_t \widehat{\Phi}_t^\tau = \mathrm{Diag}\widehat{\Phi}_t - \widehat{\Phi}_t \widehat{\Phi}_t^\tau.$$

Then, by (2.9) and (2.10), we can conclude that

$$\begin{aligned} E\widetilde{\Phi}_t^\tau \widetilde{\Phi}_t = {} & E\widetilde{\Phi}_0^\tau \widetilde{\Phi}_0 + 2E\int_0^t \widetilde{\Phi}_s^\tau \Pi \widetilde{\Phi}_s ds - 2E\int_0^t \Phi_s^\tau \Pi \Phi_s ds \\ & -E\int_0^t \mathrm{Tr}\left([\widehat{\Phi}_s \widehat{\Phi}_s^\tau - \mathrm{Diag}\widehat{\Phi}_s](H_s^\tau H_s)[\widehat{\Phi}_s \widehat{\Phi}_s^\tau - \mathrm{Diag}\widehat{\Phi}_s]\right) ds, \end{aligned}$$

i.e., (2.6) holds. □

COROLLARY 2.1. (2.6) implies that

$$E\int_0^t \sum_{i=1}^N [\widehat{\Phi}_s(i)]^2 \|A_{\widehat{\Phi}_s} x_s + B_{\widehat{\Phi}_s} u_s - A_i x_s - B_i u_s\|^2 ds \le 1 + 4\|\Pi\| t, \tag{2.11}$$

where $\widehat{\Phi}_t$ and $\widehat{\Phi}_t(i)$ are defined in (2.1), and

$$A_{\widehat{\Phi}_s} = \sum_{i=1}^N \widehat{\Phi}_s(i) A_i, \qquad B_{\widehat{\Phi}_s} = \sum_{i=1}^N \widehat{\Phi}_s(i) B_i. \tag{2.12}$$

Proof. Let

$$H_{t,i} = A_i x_t + B_i u_t \quad \text{and} \quad H_{t,\widehat{\Phi}_t} = A_{\widehat{\Phi}_t} x_t + B_{\widehat{\Phi}_t} u_t. \tag{2.13}$$

Then, by (2.1)–(2.3), we have

$$\begin{aligned} H_t[\widehat{\Phi}_t \widehat{\Phi}_t^\tau - \mathrm{Diag}\widehat{\Phi}_t] &= [H_{t,\widehat{\Phi}_t} \widehat{\Phi}_t(1), \ldots, H_{t,\widehat{\Phi}_t} \widehat{\Phi}_t(N)] - H_t \mathrm{Diag}\widehat{\Phi}_t \\ &= [(H_{t,\widehat{\Phi}_t} - H_{t,1})\widehat{\Phi}_t(1), \ldots, (H_{t,\widehat{\Phi}_t} - H_{t,N})\widehat{\Phi}_t(N)]. \end{aligned}$$

Thus, by (2.13) and (2.6) we get

$$\begin{aligned} & E\int_0^t \sum_{i=1}^N [\widehat{\Phi}_s(i)]^2 \|A_{\widehat{\Phi}_s} x_s + B_{\widehat{\Phi}_s} u_s - A_i x_s - B_i u_s\|^2 ds \\ & = E\int_0^t \sum_{i=1}^N [\widehat{\Phi}_s(i)]^2 \|H_{s,\widehat{\Phi}_s} - H_{s,i}\|^2 ds \\ & = E\int_0^t \mathrm{Tr}\left([\widehat{\Phi}_s \widehat{\Phi}_s^\tau - \mathrm{Diag}\widehat{\Phi}_s](H_s^\tau H_s)[\widehat{\Phi}_s \widehat{\Phi}_s^\tau - \mathrm{Diag}\widehat{\Phi}_s]\right) ds \\ & \le E\widetilde{\Phi}_0^\tau \widetilde{\Phi}_0 + 2E\int_0^t \widetilde{\Phi}_s^\tau \Pi \widetilde{\Phi}_s ds - 2E\int_0^t \Phi_s^\tau \Pi \Phi_s ds \\ & \le 1 + 4\|\Pi\| t, \end{aligned} \tag{2.14}$$

where we have used the fact that $E\|\widetilde{\Phi}_t\|^2 = 1 - E\|\widehat{\Phi}_t\|^2 \le 1$ and $\|\Phi_t\|^2 = 1$ for $t \ge 0$ in order to get the last inequality.

This completes the proof of Corollary 2.1. □

COROLLARY 2.2. *For any constant $\eta > 0$, we have*

$$E\int_0^T \|x_t\|^2 \sum_{i=1}^N [\widehat{\Phi}_t(i)]^2 \|A_{\widehat{\Phi}_t} x_s + B_{\widehat{\Phi}_t} u_s - A_i x_t - B_i u_t\|^2 dt$$

$$\le E\|x_0\|^2 + (\eta + 2\|\Pi\|)E\int_0^T \|x_t\|^2 dt + 2E\int_0^T |x_t^\tau H_t \widehat{\Phi}_t| dt$$

$$+4\eta^{-1}(1 + 4\|\Pi\|T) + NT. \tag{2.15}$$

Proof. From Ito's formula and (2.4) it follows that

$$d(\widehat{\Phi}_t^\tau \widehat{\Phi}_t) = 2\widehat{\Phi}_t^\tau \Pi \widehat{\Phi}_t dt + 2\widehat{\Phi}_t^\tau \left(\mathrm{Diag}\widehat{\Phi}_t - \widehat{\Phi}_t\widehat{\Phi}_t^\tau\right) H_t^\tau d\overline{w}_t$$
$$+\mathrm{Tr}\left[\left(\mathrm{Diag}\widehat{\Phi}_t - \widehat{\Phi}_t\widehat{\Phi}_t^\tau\right) H_t^\tau H_t \left(\mathrm{Diag}\widehat{\Phi}_t - \widehat{\Phi}_t\widehat{\Phi}_t^\tau\right)\right] dt,$$

and from (2.5),

$$d(x_t^\tau x_t) = 2x_t^\tau H_t \widehat{\Phi}_t dt + 2x_t^\tau d\overline{w}_t + N dt.$$

Therefore, by Ito's formula we have

$$d[(1 - \widehat{\Phi}_t^\tau \widehat{\Phi}_t) x_t^\tau x_t]$$
$$= -x_t^\tau x_t \mathrm{Tr}\left[\left(\mathrm{Diag}\widehat{\Phi}_t - \widehat{\Phi}_t\widehat{\Phi}_t^\tau\right) H_t^\tau H_t \left(\mathrm{Diag}\widehat{\Phi}_t - \widehat{\Phi}_t\widehat{\Phi}_t^\tau\right)\right] dt$$
$$-2x_t^\tau x_t \widehat{\Phi}_t^\tau \Pi \widehat{\Phi}_t dt + 2(1 - \widehat{\Phi}_t^\tau \widehat{\Phi}_t) x_t^\tau H_t \widehat{\Phi}_t dt + N(1 - \widehat{\Phi}_t^\tau \widehat{\Phi}_t) dt$$
$$-4\widehat{\Phi}_t^\tau \left(\mathrm{Diag}\widehat{\Phi}_t - \widehat{\Phi}_t\widehat{\Phi}_t^\tau\right) H_t^\tau x_t dt + 2(1 - \widehat{\Phi}_t^\tau \widehat{\Phi}_t) x_t^\tau d\overline{w}_t$$
$$-2x_t^\tau x_t \widehat{\Phi}_t^\tau \left(\mathrm{Diag}\widehat{\Phi}_t - \widehat{\Phi}_t\widehat{\Phi}_t^\tau\right) H_t^\tau d\overline{w}_t.$$

Taking expectations of both sides, and noticing $0 < \|\widehat{\Phi}_t\| \le 1$, $(1 - \widehat{\Phi}_t^\tau \widehat{\Phi}_t) x_t^\tau x_t \ge 0$, and $4ab \le 4\eta^{-1}a^2 + \eta b^2$ (for all $a, b \ge 0, \eta > 0$) we get that for any fixed constant $\eta > 0$

$$E\int_0^T \|x_t\|^2 \mathrm{Tr}\left[\left(\mathrm{Diag}\widehat{\Phi}_t - \widehat{\Phi}_t\widehat{\Phi}_t^\tau\right) H_t^\tau H_t \left(\mathrm{Diag}\widehat{\Phi}_t - \widehat{\Phi}_t\widehat{\Phi}_t^\tau\right)\right] dt$$
$$\le E\|x_0\|^2 + 2\|\Pi\|E\int_0^T \|x_t\|^2 dt + 2E\int_0^T |x_t^\tau H_t \widehat{\Phi}_t| dt + NT$$
$$+4E\int_0^T \left\|\left(\mathrm{Diag}\widehat{\Phi}_t - \widehat{\Phi}_t\widehat{\Phi}_t^\tau\right) H_t^\tau x_t\right\| dt$$
$$\le E\|x_0\|^2 + (\eta + 2\|\Pi\|)E\int_0^T \|x_t\|^2 dt$$
$$+2E\int_0^T |x_t^\tau H_t \widehat{\Phi}_t| dt + NT$$
$$+4\eta^{-1}E\int_0^T \mathrm{Tr}\left[\left(\mathrm{Diag}\widehat{\Phi}_t - \widehat{\Phi}_t\widehat{\Phi}_t^\tau\right) H_t^\tau H_t \left(\mathrm{Diag}\widehat{\Phi}_t - \widehat{\Phi}_t\widehat{\Phi}_t^\tau\right)\right] dt,$$

which, together with (2.14), leads to

$$\begin{aligned} E\int_0^T &\|x_t\|^2\mathrm{Tr}\left[\left(\mathrm{Diag}\widehat{\Phi}_t-\widehat{\Phi}_t\widehat{\Phi}_t^\tau\right)H_t^\tau H_t\left(\mathrm{Diag}\widehat{\Phi}_t-\widehat{\Phi}_t\widehat{\Phi}_t^\tau\right)\right]dt \\ &\le E\|x_0\|^2+(\eta+2\|\Pi\|)E\int_0^T\|x_t\|^2dt+2E\int_0^T|x_t^\tau H_t\widehat{\Phi}_t|dt \\ &\quad +4\eta^{-1}(1+4\|\Pi\|T)+NT, \quad \forall\eta>0, \end{aligned}$$

i.e., (2.15) holds. □

3. Quadratic index-based adaptive control. The following lemma is to be found in Caines and Nassiri-Toussi (1991).

LEMMA 3.1. *Let the Markov process X_t satisfy the following regular Ito stochastic differential equation:*

$$dX_t=b_t(X_t)dt+G_t(X_t)dw_t. \tag{3.1}$$

Furthermore, assume that there exist a $C^1(\mathbb{R}^+)\times C^2(\mathbb{R}^n)$ nonnegative function $V_\cdot(\cdot)$, a positive real number α_0, and a nonnegative function k_t, such that

$$\frac{\partial V_t(x)}{\partial t}+\mathcal{A}V_t(x)\le-\alpha_0\|x\|^2+k_t, \quad \forall x\in\mathbb{R}^n, \quad \forall t\ge0,$$

where $\mathcal{A}$ is the infinitesimal generator of (3.1).

Then, if

$$\limsup_{t\to\infty}\frac{1}{t}E\int_0^t k_sds<\infty \quad\text{and}\quad E[V_0(X_0)]<\infty,$$

$$\limsup_{t\to\infty}\frac{1}{t}E\int_0^t\|X_s\|^2ds\le\limsup_{t\to\infty}\frac{1}{\alpha_0 t}E\int_0^t k_sds<\infty. \tag{3.2}$$

Proof. By (3.1) and Ito's formula, we know that $dV_t(X_t)$ satisfies the following equality:

$$dV_t(X_t)=\left(\frac{\partial V_t(x)}{\partial t}+\mathcal{A}V_t(x)\right)dt+\frac{\partial V_t(x)}{\partial x}G_t(X_t)dw_t.$$

With the assumptions on $V_t(X_t)$, this results in

$$V_t(X_t)\le V_0(X_0)-\alpha_0\int_0^t\|X_s\|^2ds+\int_0^t k_sds+\int_0^t\frac{\partial V_s(x)}{\partial x}G_s(X_s)dw_s.$$

Taking the expectation of both sides of this inequality we get

$$E[V_t(X_t)]-E[V_0(X_0)]\le-\alpha_0E\int_0^t\|X_s\|^2ds+E\int_0^t k_sds.$$

This, combined with the positiveness of V_t, gives the desired result (3.2). □

It is well known that if (A_α,B_α) is controllable and (A_α,C) is observable (with $C^\tau C=Q$), then for all $S>0$ the following Riccati equation has a unique, positive definite solution P_α:

$$P_\alpha A_\alpha+A_\alpha^\tau P_\alpha-P_\alpha B_\alpha S^{-1}B_\alpha^\tau P_\alpha+Q=0. \tag{3.3}$$

LEMMA 3.2. *Suppose that (A_α, B_α) is controllable and (A_α, C) is observable (with $C^\tau C = Q$). If A_α and B_α are continuous or i-times differentiable with respect to α in an interval $[\alpha_*, \alpha^*]$, then so is the solution P_α.*

Proof. From Martensson (1971) we see that the solution P_α can actually be expressed in the following form:

$$P_\alpha = Y_\alpha X_\alpha^{-1}, \quad \text{for all } \alpha \in [\alpha_*, \alpha^*]$$

where the columns of the composed matrix $\left[\begin{smallmatrix} X_\alpha \\ Y_\alpha \end{smallmatrix}\right]$ are eigenvectors or generalized eigenvectors of matrix

$$\Gamma_\alpha \triangleq \begin{bmatrix} A_\alpha & -B_\alpha S^{-1} B_\alpha^\tau \\ -Q & -A_\alpha^\tau \end{bmatrix}.$$

Now, the eigenvectors (respectively, generalized eigenvectors) of a matrix are (respectively, may be chosen to be) continuous functions of its elements. Thus, if A_α and B_α are continuous with respect to α, then Y_α, X_α, and hence P_α are continuous with respect to α.

Similarly, if A_α and B_α are i-times differentiable with respect to α, then P_α is i-times differentiable with respect to α. □

We define the adaptive control law via the certainty equivalence principle and the following quadratic index:

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t (x_s^\tau Q x_s + u_s^\tau S u_s) ds.$$

Hence, we will use the following adaptive control law:

$$u_t = -S^{-1} B_{\widehat{\Phi}_t}^\tau P_{\widehat{\Phi}_t} x_t, \tag{3.4}$$

where $\widehat{\Phi}_t$ is a solution of (2.4), and $P_{\widehat{\Phi}_t}$ is a solution of (3.3) with A_α and B_α replaced by $A_{\widehat{\Phi}_t}$ and $B_{\widehat{\Phi}_t}$, respectively.

Let $\Pi(i)$ denote the ith row of matrix Π and

$$\varepsilon = \sup_{\widehat{\Phi}_t \in \mathcal{D}} \max_{i,j=1,\ldots,N} \left\{ \left\| \frac{\partial^2 \left(P_{\widehat{\Phi}_t}\right)}{\partial \widehat{\Phi}_t(i) \partial \widehat{\Phi}_t(j)} \right\| \right\}, \tag{3.5}$$

$$c_1 = \sup_{\widehat{\Phi}_t \in \mathcal{D}} \left\{ \| A_{\widehat{\Phi}_t} - B_{\widehat{\Phi}_t} S^{-1} B_{\widehat{\Phi}_t}^\tau P_{\widehat{\Phi}_t} \| \right\}, \tag{3.6}$$

$$c_2 = \sup_{\widehat{\Phi}_t \in \mathcal{D}} \max_{i=1,\ldots,N} \left\{ \left\| \frac{\partial \left(P_{\widehat{\Phi}_t}\right)}{\partial \widehat{\Phi}_t(i)} \right\| \right\}, \tag{3.7}$$

where

$$\mathcal{D} \triangleq \left\{ \widehat{\Phi}_t : \ 0 \le \widehat{\Phi}_t(i) \le 1, \ \ i = 1, \ldots, N \ \text{ with } \sum_{i=1}^N \widehat{\Phi}_t(i) = 1 \right\}.$$

In other words, $\widehat{\Phi}_t$ ranges over the closed unit simplex $\mathcal{D}$ in $\mathbb{R}^N$.

The closed-loop system referred to in the statement of the main result below is given by the system and parameter process equations (1.1), (1.2), the filter equations (2.4), (2.5), and the Riccati and feedback equations (3.3), (3.4).

THEOREM 3.1. *Suppose that $(A_{\widehat{\Phi}_t}, B_{\widehat{\Phi}_t})$ is controllable for all $\widehat{\Phi}_t$ in the closed unit simplex $\mathcal{D}$ and that for some appropriate positive matrix S, the unique solution $P_{\widehat{\Phi}_t}$ to* (3.3) *combined with the matrix Π in* (1.2) *satisfies*

$$\|\Pi\|\varepsilon + \varepsilon c_1 < \frac{1}{4N}, \qquad c_2 \sum_{i=1}^{N} \|\Pi(i)\| < \frac{1}{2}. \tag{3.8}$$

Then, under the adaptive control law (3.4) *with $\widehat{\Phi}_t$ a solution of* (2.4), *the closed-loop system has a unique strong solution $\{x_t,\ \widehat{\Phi}_t,\ t \geq 0\}$, and is stabilized in the following average sense:*

$$\limsup_{T\to\infty} \frac{1}{T} E \int_0^T (\|x_t\|^2 + \|u_t\|^2) dt < \infty.$$

To prove Theorem 3.1, we introduce some notation following Guo (1993). For any fixed positive number K, denote by $\mathcal{C}_K^{n+N}$ the space of $I\!R^{n+N}$-valued continuous functions on the interval $[0, K]$. When $g = \{g_t\}_{0\leq t\leq K}$ is a $\mathcal{C}_K^{n+N}$ process, we set $\|g\|_{[0,K]} = \max_{0\leq t\leq K} \|g_t\|$.

Proof. First of all, we show that the closed-loop system has a solution $\{x_t,\ \widehat{\Phi}_t,\ t \geq 0\}$. Let

(3.9)

$$z_t = \begin{bmatrix} x_t \\ \widehat{\Phi}_t \end{bmatrix},$$

(3.10)

$$a(z_t) = \begin{bmatrix} (A(\theta_t) - B(\theta_t)S^{-1}B^{\tau}_{\widehat{\Phi}_t} P_{\widehat{\Phi}_t})x_t \\ \Pi\widehat{\Phi}_t + \left(\mathrm{Diag}\widehat{\Phi}_t - \widehat{\Phi}_t\widehat{\Phi}_t^{\tau}\right) H_t^{\tau} \left[(A(\theta_t) - B(\theta_t)S^{-1}B^{\tau}_{\widehat{\Phi}_t} P_{\widehat{\Phi}_t})x_t - H_t\widehat{\Phi}_t\right] \end{bmatrix},$$

(3.11)

$$b(z_t) = \begin{bmatrix} 1 \\ \left(\mathrm{Diag}\widehat{\Phi}_t - \widehat{\Phi}_t\widehat{\Phi}_t^{\tau}\right) H_t^{\tau} \end{bmatrix}.$$

Then from (1.1), (2.4), and (3.4) the closed-loop system can be rewritten in the following form:

$$dz_t = a(z_t)dt + b(z_t)dw_t. \tag{3.12}$$

Obviously, it follows from (2.12) that $A_{\widehat{\Phi}_t}$ and $B_{\widehat{\Phi}_t}$ are differentiable with respect to each component of $\widehat{\Phi}_t$. This combined with Lemma 3.2 implies that $P_{\widehat{\Phi}_t}$ is continuous and bounded on $\mathcal{D}$, since $\mathcal{D}$ is a compact set. Thus, by (3.9)–(3.11), we can conclude that for any fixed $\Delta > 0$, there exists a constant $L^{(\Delta)}$ such that

$$\left[\|a(g_t) - a(h_t)\|^2 + \|b(g_t) - b(h_t)\|^2\right] 1\!\!1_{\{\|g\|_{[0,K]}\leq\Delta,\ \|h\|_{[0,K]}\leq\Delta\}} \leq L^{(\Delta)}\|g_t - h_t\|^2$$

and

$$\left[\|a(g_t)\|^2 + \|b(g_t)\|^2\right] 1\!\!1_{\{\|g\|_{[0,K]}\leq\Delta\}} \leq L^{(\Delta)}(1 + \|g_t\|^2),$$

where $\mathbb{1}_{\{\cdot\}}$ is the indicator function of the set $\{\cdot\}$.

Therefore, by Lemma 2.2 of Guo (1993) we know that there is an $\mathcal{F}_t$-time $\sigma_K > 0$ such that (3.12) has a unique strong solution $z_t(\omega)$ on $\{\omega,\ t:\ t < \sigma_K(\omega)\}$, and

$$\sup_{t<\sigma_K(\omega)} \|z_t(\omega)\| = \infty \quad \text{a.s. on } \mathcal{G} \triangleq \{\omega:\ \sigma_K(\omega) < K\}. \tag{3.13}$$

We now prove $\sigma_K(\omega) = K$ a.s., i.e., $P(\mathcal{G}) = 0$.

Substituting (3.4) into (1.1) results in

$$dx_t = [A(\theta_t) - B(\theta_t)S^{-1}B^\tau_{\widehat{\Phi}_t} P_{\widehat{\Phi}_t}]x_t dt + dw_t,$$

which together with Ito's formula leads to

$$\begin{aligned} \|x_t\|^2 = &\int_0^t x_s^\tau \left([A(\theta_s) - B(\theta_s)S^{-1}B^\tau_{\widehat{\Phi}_s} P_{\widehat{\Phi}_s}]^\tau + [A(\theta_s) - B(\theta_s)S^{-1}B^\tau_{\widehat{\Phi}_s} P_{\widehat{\Phi}_s}]\right) x_s ds \\ &+\|x_0\|^2 + 2\int_0^t x_s^\tau dw_s + nt. \end{aligned} \tag{3.14}$$

Notice that, by Lemma 3.2, $\alpha_1 \triangleq 2\sup_{s\geq 0, \widehat{\Phi}_s \in \mathcal{D}} \|A(\theta_s) - B(\theta_s)S^{-1}B^\tau_{\widehat{\Phi}_s} P_{\widehat{\Phi}_s}\| < \infty$ a.s., and that by Lemma 4 of Christopeit (1986) there is a random constant $0 < \alpha_2(\omega) < \infty$ a.s. such that

$$2\left|\int_0^t x_s^\tau dw_s\right| \leq \alpha_2(\omega)\int_0^t \|x_s\|^2 ds + \alpha_2(\omega), \quad \forall t \geq 0.$$

By (3.14) we get

$$\|x_t\|^2 \leq (\|x_0\|^2 + nt + \alpha_2(\omega)) + (\alpha_1 + \alpha_2(\omega))\int_0^t \|x_s\|^2 ds.$$

Thus, by the Bellman–Grownwall lemma (see e.g., Desoer and Vidyasagar (1975)) we have

$$\begin{aligned} \|x_t\|^2 &\leq \|x_0\|^2 + nt + \alpha_2(\omega) + (\alpha_1 + \alpha_2(\omega))\int_0^t (\|x_0\|^2 + \lambda + \alpha_2(\omega))e^{(\alpha_1+\alpha_2(\omega))(t-\lambda)} d\lambda \\ &\leq (\|x_0\|^2 + nt + \alpha_2(\omega))e^{(\alpha_1+\alpha_2(\omega))t}, \quad \forall t \geq 0. \end{aligned} \tag{3.15}$$

If $P(\mathcal{G}) > 0$, then by (3.15) and the fact that $\|\widehat{\Phi}_t\| \leq 1$ we see that

$$\begin{aligned} \sup_{0\leq t<\sigma_K(\omega)} \|z_t(\omega)\|^2 &\leq 2 + 2\sup_{0\leq t<\sigma_K(\omega)} \|x_t(\omega)\|^2 \\ &\leq (\|x_0\|^2 + n\sigma_K(\omega) + \alpha_2(\omega))e^{(\alpha_1+\alpha_2(\omega))\sigma_K(\omega)} < \infty \quad \text{a.s. on } \mathcal{G}, \end{aligned}$$

contradicting (3.13) and $P(\mathcal{G}) > 0$.

Noting that K can be any positive number, we see that the closed-loop system (3.12) has a unique strong solution $z_t(\omega)$ on any finite time interval.

We now prove the stability of the closed-loop system.

Let $\overline{A}_{\widehat{\Phi}_t} = A_{\widehat{\Phi}_t} - B_{\widehat{\Phi}_t} S^{-1} B^\tau_{\widehat{\Phi}_t} P_{\widehat{\Phi}_t}$ and $\overline{A}_{\widehat{\Phi}_t}(i) = A_i - B_i S^{-1} B^\tau_{\widehat{\Phi}_t} P_{\widehat{\Phi}_t}$. Then from (2.5) and (2.12) it follows that

$$dx_t = H_t \widehat{\Phi}_t dt + d\overline{w}_t = \overline{A}_{\widehat{\Phi}_t} x_t dt + d\overline{w}_t. \tag{3.16}$$

Applying the general Ito formula to $V(x_t) = x_t^\tau P_{\widehat{\Phi}_t} x_t$, and employing (3.3), (2.4), and (2.5), we have the following inequalities (see, e.g., Caines and Nassiri-Toussi (1991)):

$$\begin{aligned}
\mathcal{A}V(x_t) &= -x_t^\tau x_t - x_t^\tau P_{\widehat{\Phi}_t} B_{\widehat{\Phi}_t} S^{-1} B^\tau_{\widehat{\Phi}_t} P_{\widehat{\Phi}_t} x_t \\
&\quad + x_t^\tau \sum_{i=1}^N \Pi(i) \widehat{\Phi}_t \frac{\partial \left(P_{\widehat{\Phi}_t}\right)}{\partial \widehat{\Phi}_t(i)} x_t + \mathrm{Tr} P_{\widehat{\Phi}_t} \\
&\quad + x_t^\tau \sum_{i=1}^N \widehat{\Phi}_t(i) \frac{\partial \left(P_{\widehat{\Phi}_t}\right)}{\partial \widehat{\Phi}_t(i)} (\overline{A}_{\widehat{\Phi}_t} - \overline{A}_{\widehat{\Phi}_t}(i)) x_t \\
&\quad + x_t^\tau \sum_{i=1}^N \widehat{\Phi}_t(i) (\overline{A}_{\widehat{\Phi}_t} - \overline{A}_{\widehat{\Phi}_t}(i))^\tau \frac{\partial \left(P_{\widehat{\Phi}_t}\right)}{\partial \widehat{\Phi}_t(i)} x_t \\
&\quad + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \widehat{\Phi}_t(i) \widehat{\Phi}_t(j) x_t^\tau \frac{\partial^2 \left(P_{\widehat{\Phi}_t}\right)}{\partial \widehat{\Phi}_t(i) \partial \widehat{\Phi}_t(j)} x_t \\
&\qquad \cdot x_t^\tau (\overline{A}_{\widehat{\Phi}_t} - \overline{A}_{\widehat{\Phi}_t}(i))^\tau (\overline{A}_{\widehat{\Phi}_t} - \overline{A}_{\widehat{\Phi}_t}(i)) x_t \\
&\le -x_t^\tau x_t + c_2 \sum_{i=1}^N \|\Pi(i)\| \|x_t\|^2 + \mathrm{Tr} P_{\widehat{\Phi}_t} \\
&\quad + 2c_2 \|x_t\| \sum_{i=1}^N \widehat{\Phi}_t(i) \|(\overline{A}_{\widehat{\Phi}_t} - \overline{A}_{\widehat{\Phi}_t}(i)) x_t\| \\
&\quad + \frac{\varepsilon}{2} \|x_t\|^2 \left[\sum_{i=1}^N \widehat{\Phi}_t(i) \|(\overline{A}_{\widehat{\Phi}_t} - \overline{A}_{\widehat{\Phi}_t}(i)) x_t\| \right]^2 \\
&\le -\left(\frac{3}{4} - c_2 \sum_{i=1}^N \|\Pi(i)\| \right) \|x_t\|^2 + \mathrm{Tr} P_{\widehat{\Phi}_t} \\
&\quad + 4N c_2^2 \sum_{i=1}^N [\widehat{\Phi}_t(i)]^2 \|(\overline{A}_{\widehat{\Phi}_t} - \overline{A}_{\widehat{\Phi}_t}(i)) x_t\|^2 \\
&\quad + \frac{N\varepsilon}{2} \|x_t\|^2 \sum_{i=1}^N [\widehat{\Phi}_t(i)]^2 \|(\overline{A}_{\widehat{\Phi}_t} - \overline{A}_{\widehat{\Phi}_t}(i)) x_t\|^2,
\end{aligned}$$

where we have used the sum of squares bound $2ab \le 1/4a^2 + 4b^2$ and a standard sum of squares bound to obtain the last inequality above and where ε and c_2 are given by (3.5) and (3.7), respectively.

By the second inequality of condition (3.8), we see

$$\beta \triangleq \frac{3}{4} - c_2 \sum_{i=1}^N \|\Pi(i)\| > \frac{1}{4} > 0,$$

and hence, by Lemma 3.1, we get

$$
\begin{aligned}
&\limsup_{T\to\infty} \frac{1}{T} E \int_0^T \|x_t\|^2 dt \\
&\quad \le \limsup_{T\to\infty} \frac{1}{\beta T} E \int_0^T \mathrm{Tr} P_{\widehat{\Phi}_t} dt \\
&\qquad + \limsup_{T\to\infty} \frac{4Nc_2^2}{\beta T} E \int_0^T \sum_{i=1}^N [\widehat{\Phi}_t(i)]^2 \|(\overline{A}_{\widehat{\Phi}_t} - \overline{A}_{\widehat{\Phi}_t}(i)) x_t\|^2 dt \\
&\qquad + \limsup_{T\to\infty} \frac{N\varepsilon}{2\beta T} E \int_0^T \|x_t\|^2 \sum_{i=1}^N [\widehat{\Phi}_t(i)]^2 \|(\overline{A}_{\widehat{\Phi}_t} - \overline{A}_{\widehat{\Phi}_t}(i)) x_t\|^2 dt.
\end{aligned}
\tag{3.17}
$$

By (3.4), i.e., $u_t = -S^{-1} B^\tau_{\widehat{\Phi}_t} P_{\widehat{\Phi}_t} x_t$, we have

$$
[\overline{A}_{\widehat{\Phi}_t} - \overline{A}_{\widehat{\Phi}_t}(i)] x_t = A_{\widehat{\Phi}_t} x_t + B_{\widehat{\Phi}_t} u_t - A_i x_t - B_i u_t.
$$

Thus, from (3.17) and Corollaries 2.1 and 2.2 it follows that for any fixed $\eta > 0$,

$$
\begin{aligned}
&\limsup_{T\to\infty} \frac{1}{T} E \int_0^T \|x_t\|^2 dt \\
&\quad \le \limsup_{T\to\infty} \frac{1}{\beta T} E \int_0^T \mathrm{Tr} P_{\widehat{\Phi}_t} dt \\
&\qquad + \left\{16N\|\Pi\|\beta^{-1} c_2^2 + N\varepsilon(16\|\Pi\|\eta^{-1} + N)(2\beta)^{-1}\right\} \\
&\qquad + \limsup_{T\to\infty} \frac{N(\eta + 2\|\Pi\|)\varepsilon}{2\beta T} E \int_0^T \|x_t\|^2 dt \\
&\qquad + \limsup_{T\to\infty} \frac{N\varepsilon}{\beta T} E \int_0^T |x_t^\tau H_t \widehat{\Phi}_t| dt.
\end{aligned}
\tag{3.18}
$$

It is easy to see that

$$
|x_t^\tau H_t \widehat{\Phi}_t| = |x_t^\tau \overline{A}_{\widehat{\Phi}_t} x_t| \le c_1 \|x_t\|^2, \tag{3.19}
$$

where c_1 is defined in (3.6).

Substituting (3.19) into (3.18) we get that for all $\eta > 0$,

$$
\begin{aligned}
&\limsup_{T\to\infty} \frac{1}{T} E \int_0^T \|x_t\|^2 dt \\
&\quad \le \limsup_{T\to\infty} \frac{1}{\beta T} E \int_0^T \mathrm{Tr} P_{\widehat{\Phi}_t} dt \\
&\qquad + \left\{16N\|\Pi\|\beta^{-1} c_2^2 + N\varepsilon(16\|\Pi\|\eta^{-1} + N)(2\beta)^{-1}\right\} \\
&\qquad + \limsup_{T\to\infty} \frac{N(\eta + 2\|\Pi\|)\varepsilon + 2N\varepsilon c_1}{2\beta T} E \int_0^T \|x_t\|^2 dt.
\end{aligned}
\tag{3.20}
$$

Notice that (3.8) implies

$$
N\beta^{-1}(\|\Pi\|\varepsilon + \varepsilon c_1) < 1.
$$

So we can fix a constant $\eta > 0$ at such a value that

$$\frac{N(\eta + 2\|\Pi\|)\varepsilon + 2N\varepsilon c_1}{2\beta} < 1. \tag{3.21}$$

Recalling that $P_{\widehat{\Phi}_t}$ is bounded on $\mathcal{D}$, we get

$$\limsup_{T\to\infty} \frac{1}{T} E \int_0^T \mathrm{Tr} P_{\widehat{\Phi}_t} dt < \infty,$$

and hence, by (3.21) and (3.20) we have

$$\limsup_{T\to\infty} \frac{1}{T} E \int_0^T \|x_t\|^2 dt < \infty,$$

which together with (3.4) results in

$$\limsup_{T\to\infty} \frac{1}{T} E \int_0^T \|u_t\|^2 dt < \infty.$$

Therefore, Theorem 3.1 is true. □

4. An example. In this section, we present an example to demonstrate that the conditions of Theorem 3.1 are verifiable in certain nontrivial cases.

EXAMPLE 4.1. If system (1.1) is such that $n = 2$, $m = 1$, $B_1 = B_2 = \cdots = B_N = \left[\begin{smallmatrix} 0 \\ b \end{smallmatrix}\right]$ with

$$b \neq 0 \quad \text{and} \quad A_i = \begin{bmatrix} 0 & 1 \\ 0 & -a_i \end{bmatrix}$$

for a_i distinct, $i = 1, \ldots, N$, then (i) $(A_{\widehat{\Phi}_t}, B_{\widehat{\Phi}_t})$ is controllable for all $\widehat{\Phi}_t$ in the closed unit simplex, and (ii) condition (3.8) and the conclusion of Theorem 3.1 are true when the parameter S in the control Riccati equation (4.2) for $P_{\widehat{\Phi}_t}$ is sufficiently small.

Proof. The truth of (i) is evident. Concerning (ii) set

$$a_{\widehat{\Phi}_t} = \sum_{i=0}^{N} \widehat{\Phi}_t(i) a_i \quad \text{and} \quad P_{\widehat{\Phi}_t} = \begin{bmatrix} P_{\widehat{\Phi}_t}(1,1) & P_{\widehat{\Phi}_t}(1,2) \\ P_{\widehat{\Phi}_t}(1,2) & P_{\widehat{\Phi}_t}(2,2) \end{bmatrix}. \tag{4.1}$$

Then the algebraic Riccati equation (3.3) becomes

$$\begin{aligned} &\begin{bmatrix} P_{\widehat{\Phi}_t}(1,1) & P_{\widehat{\Phi}_t}(1,2) \\ P_{\widehat{\Phi}_t}(1,2) & P_{\widehat{\Phi}_t}(2,2) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -a_{\widehat{\Phi}_t} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & -a_{\widehat{\Phi}_t} \end{bmatrix} \begin{bmatrix} P_{\widehat{\Phi}_t}(1,1) & P_{\widehat{\Phi}_t}(1,2) \\ P_{\widehat{\Phi}_t}(1,2) & P_{\widehat{\Phi}_t}(2,2) \end{bmatrix} \\ &- \begin{bmatrix} P_{\widehat{\Phi}_t}(1,1) & P_{\widehat{\Phi}_t}(1,2) \\ P_{\widehat{\Phi}_t}(1,2) & P_{\widehat{\Phi}_t}(2,2) \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & S^{-1}b^2 \end{bmatrix} \begin{bmatrix} P_{\widehat{\Phi}_t}(1,1) & P_{\widehat{\Phi}_t}(1,2) \\ P_{\widehat{\Phi}_t}(1,2) & P_{\widehat{\Phi}_t}(2,2) \end{bmatrix} + I = 0, \end{aligned} \tag{4.2}$$

which is equivalent to

$$\begin{aligned} 0 &= 1 - S^{-1}b^2 P^2_{\widehat{\Phi}_t}(1,2), \\ 0 &= P_{\widehat{\Phi}_t}(1,1) - S^{-1}b^2 P_{\widehat{\Phi}_t}(1,2) P_{\widehat{\Phi}_t}(2,2) - a_{\widehat{\Phi}_t} P_{\widehat{\Phi}_t}(1,2), \\ 0 &= P^2_{\widehat{\Phi}_t}(2,2) + 2Sb^{-2} a_{\widehat{\Phi}_t} P_{\widehat{\Phi}_t}(2,2) - Sb^{-2}(1 + 2|b|^{-1}\sqrt{S}). \end{aligned}$$

Solving this set of equations we get

$$P_{\widehat{\Phi}_t}(1,1) = |b|^{-1}\sqrt{S}\left[a^2_{\widehat{\Phi}_t} + S^{-1}b^2 + 2|b|S^{-1/2}\right]^{1/2}, \tag{4.3}$$

$$P_{\widehat{\Phi}_t}(1,2) = |b|^{-1}\sqrt{S}, \tag{4.4}$$

$$P_{\widehat{\Phi}_t}(2,2) = Sb^{-2}\left[a^2_{\widehat{\Phi}_t} + S^{-1}b^2 + 2|b|S^{-1/2}\right]^{1/2} - Sb^{-2}a_{\widehat{\Phi}_t}. \tag{4.5}$$

Hence, when S is small enough,

$$\begin{aligned}
\mathrm{Tr}P_{\widehat{\Phi}_t} &= P_{\widehat{\Phi}_t}(1,1) + P_{\widehat{\Phi}_t}(2,2)\\
&= -Sb^{-2}a_{\widehat{\Phi}_t} + (|b|^{-1}\sqrt{S} + Sb^{-2})\left[a^2_{\widehat{\Phi}_t} + S^{-1}b^2 + 2|b|S^{-1/2}\right]^{1/2}\\
&= O(S),
\end{aligned}$$

where $O(S)$ denotes a function of S satisfying $\limsup_{S\to 0}\frac{|O(S)|}{S} < \infty$.
From this it follows that

$$\limsup_{T\to\infty}\frac{1}{T}E\int_0^T \mathrm{Tr}P_{\widehat{\Phi}_t}\,dt < \infty.$$

Let $\mu = S^{-1}b^2 + 2|b|S^{-1/2}$ and $\gamma_{\widehat{\Phi}_t} = \left[a^2_{\widehat{\Phi}_t} + \mu\right]^{1/2}$. Then it is easy to see that

$$\frac{\partial\left(\gamma_{\widehat{\Phi}_t}\right)}{\partial\widehat{\Phi}_t(i)} = \left[a^2_{\widehat{\Phi}_t} + \mu\right]^{-1/2} a_{\widehat{\Phi}_t}\frac{\partial\left(a_{\widehat{\Phi}_t}\right)}{\partial\widehat{\Phi}_t(i)} = \left[a^2_{\widehat{\Phi}_t} + \mu\right]^{-1/2} a_{\widehat{\Phi}_t}a_i, \tag{4.6}$$

where $i = 1, \ldots, N$.
Furthermore,

$$\begin{aligned}
\frac{\partial^2\left(\gamma_{\widehat{\Phi}_t}\right)}{\partial\widehat{\Phi}_t(i)\partial\widehat{\Phi}_t(j)} &= \left[a^2_{\widehat{\Phi}_t} + \mu\right]^{-1/2} a_ia_j - a^2_{\widehat{\Phi}_t}\left[a^2_{\widehat{\Phi}_t} + \mu\right]^{-3/2} a_ia_j\\
&= \mu\left[a^2_{\widehat{\Phi}_t} + \mu\right]^{-3/2} a_ia_j, \quad i,\ j = 1, \ldots, N.
\end{aligned} \tag{4.7}$$

From (4.3)–(4.6) it follows that for $i = 1, \ldots,$ or N,

$$\frac{\partial\left(P_{\widehat{\Phi}_t}\right)}{\partial\widehat{\Phi}_t(i)} = \begin{bmatrix} \dfrac{a_ia_{\widehat{\Phi}_t}|b|^{-1}S^{1/2}}{\left[a^2_{\widehat{\Phi}_t} + S^{-1}b^2 + 2|b|S^{-1/2}\right]^{1/2}} & 0\\ 0 & \dfrac{a_ia_{\widehat{\Phi}_t}Sb^{-2}}{\left[a^2_{\widehat{\Phi}_t} + S^{-1}b^2 + 2|b|S^{-1/2}\right]^{1/2}} - Sb^{-2}a_i \end{bmatrix},$$

which implies that for S sufficiently small

$$c_2 \equiv c_2(S) \le c_3S, \tag{4.8}$$

where c_3 is a constant depending on a_i and b only.

Since

$$\frac{\partial^2\left(a_{\widehat{\Phi}_t}\right)}{\partial\widehat{\Phi}_t(i)\partial\widehat{\Phi}_t(j)} = 0,$$

(4.3)–(4.5) and (4.7) yield

$$\frac{\partial^2\left(P_{\widehat{\Phi}_t}\right)}{\partial\widehat{\Phi}_t(i)\partial\widehat{\Phi}_t(j)} = \begin{bmatrix} \dfrac{a_i a_j(2+|b|S^{-1/2})}{\left[a^2_{\widehat{\Phi}_t} + S^{-1}b^2 + 2|b|S^{-1/2}\right]^{3/2}} & 0 \\ 0 & \dfrac{a_i a_j(1+2|b|^{-1}S^{1/2})}{\left[a^2_{\widehat{\Phi}_t} + S^{-1}b^2 + 2|b|S^{-1/2}\right]^{3/2}} \end{bmatrix}.$$

From this we obtain that as $S \to 0$,

$$\left\|\frac{\partial^2\left(P_{\widehat{\Phi}_t}\right)}{\partial\widehat{\Phi}_t(i)\partial\widehat{\Phi}_t(j)}\right\| = |a_i a_j| S b^{-2}\left(1 + O(S^{1/2})\right),$$

which implies that as $S \to 0$,

$$\varepsilon = S \max_{i,\, j=1,\ldots,\, N} |a_i a_j| b^{-2}\left(1 + O(S^{1/2})\right). \tag{4.9}$$

From (4.3)–(4.5) it follows that

$$\begin{bmatrix} 0 & 1 \\ 0 & a_{\widehat{\Phi}_t} \end{bmatrix} - S^{-1}\begin{bmatrix} 0 \\ b \end{bmatrix}\begin{bmatrix} 0 & b \end{bmatrix}\begin{bmatrix} P_{\widehat{\Phi}_t}(1,1) & P_{\widehat{\Phi}_t}(1,2) \\ P_{\widehat{\Phi}_t}(1,2) & P_{\widehat{\Phi}_t}(2,2) \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 \\ -|b|S^{-1/2} & 2a_{\widehat{\Phi}_t} - \left[a^2_{\widehat{\Phi}_t} + S^{-1}b^2 + 2|b|S^{-1/2}\right]^{1/2} \end{bmatrix}.$$

Then we get

$$\|A_{\widehat{\Phi}_t} - S^{-1}B_{\widehat{\Phi}_t}B^{\tau}_{\widehat{\Phi}_t}P_{\widehat{\Phi}_t}\| \le 2S^{-1/2}|b|\left(1 + O(S^{1/2})\right),$$

which implies that

$$c_1 \le 2S^{-1/2}|b|\left(1 + O(S^{1/2})\right).$$

From this and (4.8), (4.9) we see that for some sufficiently small S, condition (3.8), and hence the results of Theorem 3.1, are true. □

5. Maximum likelihood-based adaptive control. Intuitively, if $\widehat{\Phi}_t$ is a good estimate of Φ_t, in some sense, then $A_{\widehat{\Phi}_t}$ and $B_{\widehat{\Phi}_t}$ are good estimates of $A(\theta_t)$ and $B(\theta_t)$. Therefore, in the last two sections, we discuss the stabilization problem of the filtered system (2.5):

$$dx_t = A_{\widehat{\Phi}_t}x_t dt + B_{\widehat{\Phi}_t}u_t dt + d\overline{w}_t \quad \text{by (2.1), (2.2), and (2.12),}$$

rather than that of system (1.1).

Let i_t be defined by

$$i_t = \arg \max_{i=1,\dots,N} \{\widehat{\Phi}_t(i)\}, \quad t \geq 0. \tag{5.1}$$

Again, if $\widehat{\Phi}_t$ is a good estimate of Φ_t, in some sense, then $A(i_t)$ and $B(i_t)$ should also be good estimates of $A(\theta_t)$ and $B(\theta_t)$. In this case, it is natural to ask whether we could find an adaptive stabilization control law for system (1.1) by only discussing the following system:

$$dx_t = A(i_t)x_t dt + B(i_t)u_t dt + d\overline{w}_t\,.$$

This section, as an application of Corollary 2.1, will answer this problem. By using the notion of a maximum likelihood estimate, we present some sufficient conditions for stabilization control of the system (1.1)–(1.2). These sufficient conditions are different from those used in §3, but similar to those introduced in Ezzine and Haddad (1989).

For simplicity of notation, for a matrix A, let

$$\mu(A) = \lambda_{\max}\left(\frac{A + A^\tau}{2}\right).$$

THEOREM 5.1. *Suppose there is a matrix $K(i)$ $(i = 1, \dots, N)$ such that*

$$\nu \stackrel{\Delta}{=} -\max_{i=1,\dots,N} \mu\left(A(i) - B(i)K(i)\right) > 0. \tag{5.2}$$

Then, under the adaptive control law $u_t = -K(i_t)x_t$, the closed-loop system has a solution $\{x_t,\, u_t,\, t \geq 0\}$, and the input and output of the closed-loop system are bounded in the following average sense:

$$\sup_{t\geq 0} \frac{1}{t+1} E \int_0^t (\|x_s\|^2 + \|u_s\|^2) ds < \infty. \tag{5.3}$$

Proof. Similar to the argument of Theorem 3.1, we see that the closed-loop system has a solution $\{x_t,\, \widehat{\Phi}_t,\, t \geq 0\}$. So, here we only need prove (5.3).

From (2.5) and (2.12) it follows that

$$\begin{aligned} dx_t &= H_t\widehat{\Phi}_t dt + d\overline{w}_t = A_{\widehat{\Phi}_t} x_t dt + B_{\widehat{\Phi}_t} u_t dt + d\overline{w}_t \\ &= A(i_t)x_t dt + B(i_t)u_t dt \\ &\quad + [A_{\widehat{\Phi}_t} x_t + B_{\widehat{\Phi}_t} u_t - A(i_t)x_t - B(i_t)u_t]dt + d\overline{w}_t, \end{aligned} \tag{5.4}$$

where i_t is given in (5.1)

Substituting $u_t = -K(i_t)x_t$ into (5.4) we get

$$\begin{aligned} dx_t &= [A(i_t) - B(i_t)K(i_t)]x_t dt \\ &\quad + [A_{\widehat{\Phi}_t} x_t + B_{\widehat{\Phi}_t} u_t - A(i_t)x_t - B(i_t)u_t]dt + d\overline{w}_t, \end{aligned} \tag{5.5}$$

which together with Ito's formula and (5.2) implies that for the ν given by (5.2)

$$\|x_t\|^2 = \|x_0\|^2 + \int_0^t x_s^\tau \left([A(i_s) - B(i_s)K(i_s)]^\tau + [A(i_s) - B(i_s)K(i_s)]\right) x_s ds$$

$$
\begin{aligned}
&+t+2\int_0^t x_s^\tau d\overline{w}_s + 2\int_0^t x_s^\tau[A_{\widehat{\Phi}_s}x_s + B_{\widehat{\Phi}_s}u_s - A(i_s)x_s - B(i_s)u_s]ds \\
&\le \|x_0\|^2 - 2\nu\int_0^t \|x_s\|^2 ds + 2\int_0^t x_s^\tau d\overline{w}_s + t \\
&+2\int_0^t x_s^\tau[A_{\widehat{\Phi}_s}x_s + B_{\widehat{\Phi}_s}u_s - A(i_s)x_s - B(i_s)u_s]ds. \qquad (5.6)
\end{aligned}
$$

Notice that

$$
\begin{aligned}
&2\int_0^t x_s^\tau[A_{\widehat{\Phi}_s}x_s + B_{\widehat{\Phi}_s}u_s - A(i_s)x_s - B(i_s)u_s]ds \\
&\le \nu\int_0^t \|x_s\|^2 ds + \nu^{-1}\int_0^t \|A_{\widehat{\Phi}_s}x_s + B_{\widehat{\Phi}_s}u_s - A(i_s)x_s - B(i_s)u_s\|^2 ds;
\end{aligned}
$$

by (5.6) we get

$$
\begin{aligned}
\|x_t\|^2 &\le \|x_0\|^2 - \nu\int_0^t \|x_s\|^2 ds + 2\int_0^t x_s^\tau d\overline{w}_s + t \\
&+\nu^{-1}\int_0^t \|A_{\widehat{\Phi}_s}x_s + B_{\widehat{\Phi}_s}u_s - A(i_s)x_s - B(i_s)u_s\|^2 ds,
\end{aligned}
$$

which implies

$$
\begin{aligned}
E\int_0^t \|x_s\|^2 ds &\le \nu^{-2}E\int_0^t \|A_{\widehat{\Phi}_s}x_s + B_{\widehat{\Phi}_s}u_s - A(i_s)x_s - B(i_s)u_s\|^2 ds \\
&+\nu^{-1}E\|x_0\|^2 + \nu^{-1}t. \qquad (5.7)
\end{aligned}
$$

By (2.1) we see that $\widehat{\Phi}_t(i) \ge 0$ for $i = 1, \dots, N$ and $t \ge 0$; further, since

$$\sum_{i=1}^N \widehat{\Phi}_t(i) = 1,$$

we have $\widehat{\Phi}_t(i_t) \ge \frac{1}{N}$. Thus, by Corollary 2.1 we get

(5.8)

$$
\begin{aligned}
&E\int_0^t \|A_{\widehat{\Phi}_s}x_s + B_{\widehat{\Phi}_s}u_s - A(i_s)x_s - B(i_s)u_s\|^2 ds \\
&\le N^2 E\int_0^t \|\widehat{\Phi}_s(i_s)]^2[A_{\widehat{\Phi}_s}x_s + B_{\widehat{\Phi}_s}u_s - A(i_s)x_s - B(i_s)u_s\|^2 ds \\
&\le N^2 E\int_0^t \sum_{i=1}^N [\widehat{\Phi}_s(i)]^2\|A_{\widehat{\Phi}_s}x_s + B_{\widehat{\Phi}_s}u_s - A(i)x_s - B(i)u_s\|^2 ds \quad (\text{since } i_s \in \{1, \dots, N\}) \\
&\le N^2(2 + 6\|\Pi\|t).
\end{aligned}
$$

Substituting this into (5.7) leads to the desired result, (5.3). □

From the definition (5.1) of i_t and $u_t = -K(i_t)x_t$ it follows that u_t may jump at any time instant t. In order to get a piecewise continuous control u_t, that is, one that has with probability 1 no accumulation points of switching times on the time axis, one can modify the definition (5.1) of i_t as follows:

$$i_t = i'_{\tau_{k-1}}, \quad \forall t \in [\tau_{k-1}, \ \tau_k), \quad \forall k = 1, \ 2, \dots, \qquad (5.9)$$

where

$$i'_t = \arg \max_{i=1,\ldots,N} \{\widehat{\Phi}_t(i)\}, \quad \forall t \geq 0, \tag{5.10}$$

$$\tau_k = \inf \left\{ t > \tau_{k-1} : \quad \widehat{\Phi}_t(i_{\tau_{k-1}}) \leq (\gamma N)^{-1} \right\} \tag{5.11}$$

with $\tau_0 = 0$, $\gamma > 1$, and $k = 1, 2, \ldots$ being positive integers.

Since the trajectories of $\widehat{\Phi}$ are continuous and $\gamma > 1$ it is evident that $K(\cdot)$ and hence u has the required piecewise continuous property.

THEOREM 5.2. *If* $\{i_t;\ t \geq 0\}$ *and* $\{\tau_k;\ k = 1, 2, \ldots\}$ *are generated from* (5.9)–(5.11) *and* $u_t \in \mathcal{F}_t^x$, *then* $\lim_{k\to\infty} \tau_k = \infty$ *a.s. and* i_t *is piecewise constant a.s. Furthermore, if condition* (5.2) *of Theorem* 5.1 *is true and the adaptive control law is chosen to be* $u_t = -K(i_t)x_t$, *then* u_t *is piecewise continuous and the input and output of the closed-loop system are bounded in the average sense* (5.3).

Proof. First, we show $\lim_{k\to\infty} \tau_k = \infty$ a.s. Noticing that

$$\max_{i=1,\ldots,N} \{\widehat{\Phi}_t(i)\} \geq N^{-1}$$

and every component of $\widehat{\Phi}_t$ is a continuous function of t, by $\gamma > 1$ we see that $\tau_k > \tau_{k-1}$. Thus, $\lim_{k\to\infty} \tau_k$ exists a.s.

If the sample set $\mathcal{S} \triangleq \{\omega : \lim_{k\to\infty} \tau_k < \infty\}$ had positive probability, i.e., $P(\mathcal{S}) > 0$, then there would exist a deterministic constant $T < \infty$ such that $\mathcal{S}_1 \triangleq \{\omega : \lim_{k\to\infty} \tau_k \leq T\}$ with positive probability, i.e., $P(\mathcal{S}_1) > 0$.

Notice that for any constant $t \geq 0$,

$$\{\omega :\ 0 < \tau_k I_{\mathcal{S}_1} \leq t\} = \{\omega :\ 0 < \tau_k \leq t\} \cap \{\omega :\ I_{\mathcal{S}_1} = 1\} \in \mathcal{F}_t^x,$$

where

$$I_{\mathcal{S}_1} = \begin{cases} 1, & \text{if } \omega \in \mathcal{S}_1, \\ 0, & \text{if } \omega \notin \mathcal{S}_1. \end{cases}$$

By $\|\widehat{\Phi}_{\tau_{k+1}} - \widehat{\Phi}_{\tau_k}\|^2 \geq N^{-2}(1-\gamma^{-1})^2 > 0$, (2.4), and (2.14) we have

$$\begin{aligned}
\infty &= \sum_{k=0}^{\infty} N^{-2}(1-\gamma^{-1})^2 P(\mathcal{S}_1) \leq E I_{\mathcal{S}_1} \sum_{k=0}^{\infty} \|\widehat{\Phi}_{\tau_{k+1}} - \widehat{\Phi}_{\tau_k}\|^2 \\
&\leq 2\|\Pi\|^2 E I_{\mathcal{S}_1} \sum_{k=0}^{\infty} (\tau_{k+1} - \tau_k)^2 + 2E I_{\mathcal{S}_1} \sum_{k=0}^{\infty} \left(\int_{\tau_k}^{\tau_{k+1}} (\mathrm{Diag}\widehat{\Phi}_s - \widehat{\Phi}_s \widehat{\Phi}_s^\tau) H_s^\tau d\overline{w}_s \right)^2 \\
&\leq 2\|\Pi\|^2 T E I_{\mathcal{S}_1} \sum_{k=0}^{\infty} (\tau_{k+1} - \tau_k) + 2E \sum_{k=0}^{\infty} \int_{\tau_k I_{\mathcal{S}_1}}^{\tau_{k+1} I_{\mathcal{S}_1}} \left\| (\mathrm{Diag}\widehat{\Phi}_s - \widehat{\Phi}_s \widehat{\Phi}_s^\tau) H_s^\tau \right\|^2 ds \\
&\leq 2\|\Pi\|^2 T^2 P(\mathcal{S}_1) + 2E \int_0^T \left\| (\mathrm{Diag}\widehat{\Phi}_s - \widehat{\Phi}_s \widehat{\Phi}_s^\tau) H_s^\tau \right\|^2 ds \\
&\leq 2\|\Pi\|^2 T^2 P(\mathcal{S}_1) + 2(2 + 6\|\Pi\| T) < \infty.
\end{aligned}$$

This contradiction means that $\lim_{k\to\infty} \tau_k = \infty$ a.s. Thus, from $\tau_k > \tau_{k-1}$ and (5.9) it follows that i_t is piecewise constant a.s.

As in Theorem 5.1, with condition (5.2) we can prove that the under-control law $u_t = -K(i_t)x_t$, with i_t given by (5.9)–(5.11), the closed-loop system has a solution

$\{x_t,\ t \geq 0\}$ a.s. and is stabilized in the average sense of (5.3); this is because the only difference between the proofs is due to (5.8), which now becomes

$$E \int_0^t \|A_{\widehat{\Phi}_s} x_s + B_{\widehat{\Phi}_s} u_s - A(i_s)x_s - B(i_s)u_s\|^2 ds \leq (\gamma N)^2(2 + 6\|\Pi\|t),$$

since, in this case, $\widehat{\Phi}_t(i_t) \geq (\gamma N)^{-1}$ for all $t \geq 0$.

Noticing that $\lim_{k\to\infty} \tau_k = \infty$ a.s. and that i_t is piecewise constant a.s., we see that the control $u_t = -K(i_t)x_t$ is almost surely defined for all $t \geq 0$ and is a piecewise continuous function of t. □

Remark 5.1. Although it is hard to say whether or not condition (5.2) is true in general cases, there exist specific situations where it is readily verified; for instance, (i) the case where $A(i)$ and $B(i)$ are scalar and $(A(i), B(i))$ is stabilizable for every $i = 1, \dots, N$, and (ii) that where $B(i)$ is invertible for $i = 1, \dots, N$, and there exists $K(i)$ such that (5.2) holds.

In fact, for case (i), $K(i)$ can be chosen as

$$K(i) = \begin{cases} [B(i)]^{-1}[1 + A(i)], & \text{if } B(i) \neq 0, \\ 0, & \text{if } B(i) = 0, \end{cases}$$

and the constant ν in (5.2) may be taken equal to the following positive quantity:

$$-\max\{\nu_i, \quad i = 1, \dots, N\},$$

where

$$\nu_i = \begin{cases} -1, & \text{if } B(i) \neq 0, \\ A(i), & \text{if } B(i) = 0. \end{cases}$$

For case (ii), $K(i)$ can be chosen as $K(i) = [B(i)]^{-1}[I + A(i)]$, which results in $\nu = 1$.

Remark 5.2. We now revisit the example given by Dufour and Bertrand (1993). In (1.1), they set $n = 2$, $m = 1$,

$$B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, A_1 = \begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}.$$

In this case the conditions of Theorem 3.1 above do not hold, but the conditions of the theorem of Dufour and Bertrand are valid.

However, for this example, the adaptive control law described in Theorem 5.1 or 5.2 is applicable, and can stabilize the closed-loop system. This is because for $K(1) = [0,\ 4]$ and $K(2) = [0,\ -1]$, we get $\nu = \frac{3-\sqrt{2}}{2} > 0$ by a straightforward manipulation. This implies that condition (5.2), and hence the conclusion of Theorems 5.1 and 5.2, are true.

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